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A necessary condition for the integrability of homogeneous Hamiltonian systems with two degrees of freedom

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Abstract

A necessary condition for the integrability of Hamiltonian systems with a two-dimensional homogeneous potential, due to Morales-Ruiz and Ramis, is extended for more general Hamiltonian systems of the form $H = T(p) + V(q)$, with homogeneous functions $T(p)$ and $V(q)$.

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1. Introduction

A Hamiltonian system with n degrees of freedom is integrable if there exist n independent first integrals in involution (Liouville integrability). For the case of two degrees of freedom in particular, the existence of a first integral which is independent of the Hamiltonian guarantees the integrability of the system. It is a fundamental and important problem to determine whether a given Hamiltonian system is integrable or not. At present, however, there are no ultimate algorithms (necessary and sufficient conditions for integrability) which provide the answer to this problem.

The Painlevé property of the solution in singularity analysis [10] is believed to be strongly related to the integrability of the system, which is known as the Painlevé conjecture. The singularity analysis has its origin in the discovery of the Kowalevski top [6] (see [1] for a brief history). Although any rigorous relation between the Painlevé property of the solution and the integrability of the system has not been established, some new integrable systems, including the Kowalevski top [6], were discovered [2, 9] using the postulate that the solution possesses the Painlevé property. These are good examples that the Painlevé property works well as a sufficient condition for integrability.

On the other hand, Morales-Ruiz and Ramis [7] gave a strong necessary condition for the integrability of Hamiltonian systems with n degrees of freedom of the form

$$H = \frac{1}{2}\boldsymbol{p}^2 + V(\boldsymbol{q}) \tag{1}$$

where $V(\boldsymbol{q})$ is a homogeneous function of integer degree. The necessary condition was obtained by the combination of the following three facts.

Fact 1. The variational equation around a straight-line solution of the system is transformed into the Gauss hypergeometric equation by a proper change of the independent variable [12].

Fact 2. If the system is integrable, then the Gauss hypergeometric equation is solvable in the sense of the differential Galois theory, i.e. the solution is obtained only by a combination of algebraic functions, quadratures and exponential of quadratures [8].

Fact 3. A necessary and sufficient condition for the solvability of the Gauss hypergeometric equations was given by Kimura [5] (Kimura's theorem).

Recently, Yoshida [14] gave a direct and independent proof of the theorem of Morales-Ruiz and Ramis [7] in the case of two degrees of freedom and justified the so-called weak Painlevé property [9] as a necessary condition for integrability for the first time. This paper shows that the proof of Yoshida [14] is easily extended for more general Hamiltonian systems of the form

$$H = T(\boldsymbol{p}) + V(\boldsymbol{q}) = T(p_1, p_2) + V(q_1, q_2) \tag{2}$$

where $T(\boldsymbol{p})$ and $V(\boldsymbol{q})$ are homogeneous functions of integer degrees m and k, respectively. Here we assume that $m \neq 0$ and $k \neq 0$. Fact 1 remains true for the generalized system (2) [13], which is summarized in section 2. Fact 2 is shown as theorem 1 in section 3, where the proof of Yoshida [14] is generalized for the system (2). Then, section 4 gives a necessary condition for the integrability with the help of Kimura's theorem, followed by some examples in section 5. In section 6, the necessary condition is rewritten in terms of the Kowalevski exponents [1, 11] in the field of singularity analysis [10]. Before closing this section, we note that all computations in this paper can be done over the field of complex numbers.

2. Derivation of the Gauss hypergeometric equation from the variational equation

Let us assume that the canonical equations of the system (2),

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \nabla T(\boldsymbol{p}) \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\nabla V(\boldsymbol{q}) \tag{3}$$

have a straight-line solution of the form

$$\boldsymbol{q} = \boldsymbol{c}Q(t) \qquad \boldsymbol{p} = \boldsymbol{c}P(t). \tag{4}$$

Here, $\boldsymbol{c} = (c_1, c_2)$ is a solution of the algebraic equations

$$\boldsymbol{c} = \nabla T(\boldsymbol{c}) = \nabla V(\boldsymbol{c}) \tag{5}$$

and $Q(t)$, $P(t)$ satisfy the differential equations

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = P^{m-1} \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = -Q^{k-1}. \tag{6}$$

Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ denote variations from the straight-line solution (4) in the direction of $\boldsymbol{q}$ and $\boldsymbol{p}$, respectively. Then, the linear variational equations of the canonical equations (3) around the straight-line solution (4) are given by

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = P^{m-2}\partial^2 T(\boldsymbol{c})\boldsymbol{\eta} \qquad \frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}t} = -Q^{k-2}\partial^2 V(\boldsymbol{c})\boldsymbol{\xi} \tag{7}$$

where

$$\partial^2 T(\boldsymbol{c}) := \left(\frac{\partial^2 T}{\partial p_i \partial p_j}\right)_{p=c} \qquad \partial^2 V(\boldsymbol{c}) := \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_{q=c} \tag{8}$$

are Hessian matrices of the functions $T(\boldsymbol{p})$, $V(\boldsymbol{q})$, evaluated at the point $\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{c}$. Since the Hessian matrices (8) are symmetric and are easily shown to be commutative, they are simultaneously diagonalizable by the change of variables $\boldsymbol{\xi} = U\boldsymbol{\xi}', \boldsymbol{\eta} = U\boldsymbol{\eta}'$ with an orthogonal matrix U. We note here that the Hessian matrices $\partial^2 T(\boldsymbol{c})$ and $\partial^2 V(\boldsymbol{c})$ have a common eigenvector $\boldsymbol{c}$, and the corresponding eigenvalues are $m-1$ and $k-1$, respectively. Then, after the orthogonal transformation, the variational equations (7) become

$$\frac{\mathrm{d}\boldsymbol{\xi}'}{\mathrm{d}t} = P^{m-2}\begin{pmatrix} \mu & 0 \\ 0 & m-1 \end{pmatrix}\boldsymbol{\eta}' \qquad \frac{\mathrm{d}\boldsymbol{\eta}'}{\mathrm{d}t} = -Q^{k-2}\begin{pmatrix} \lambda & 0 \\ 0 & k-1 \end{pmatrix}\boldsymbol{\xi}' \tag{9}$$

where μ and λ are the other eigenvalues of the Hessian matrices $\partial^2 T(\boldsymbol{c})$ and $\partial^2 V(\boldsymbol{c})$, respectively. The first components of equations (9), given by

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \mu P^{m-2}\eta \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = -\lambda Q^{k-2}\xi \tag{10}$$

describe the variation normal to the straight-line solution (4) and are called the normal variational equations (NVE). Here, for the sake of simplicity, we omit the primes and suffixes. Let us fix the energy of the straight-line motion described by (6) as

$$h = \frac{1}{m}P^m + \frac{1}{k}Q^k = \text{constant} = \frac{1}{k}. \tag{11}$$

Then, by the change of the independent variable, $t \to z$, defined by

$$z = \{Q(t)\}^k \tag{12}$$

the NVE (10) are transformed into

$$\frac{\mathrm{d}\xi}{\mathrm{d}z} = \frac{\mu}{k}z^{(1-k)/k}\left[\frac{m}{k}(1-z)\right]^{-1/m}\eta \qquad \frac{\mathrm{d}\eta}{\mathrm{d}z} = -\frac{\lambda}{k}z^{-1/k}\left[\frac{m}{k}(1-z)\right]^{(1-m)/m}\xi. \tag{13}$$

By eliminating η from these two equations, we obtain the Gauss hypergeometric equation

$$z(1-z)\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2} + [\gamma - (\alpha+\beta+1)z]\frac{\mathrm{d}\xi}{\mathrm{d}z} - \alpha\beta\xi = 0 \tag{14}$$

with the parameters

$$\alpha + \beta = \frac{1}{m} - \frac{1}{k} \qquad \alpha\beta = -\frac{\mu\lambda}{mk} \qquad \gamma = 1 - \frac{1}{k}. \tag{15}$$

This Gauss hypergeometric equation has three parameters, m, k and $\Lambda = \mu\lambda$.

3. Integrability of the system and solvability of the Gauss hypergeometric equation

The integrability of the system (2) leads to the solvability of the Gauss hypergeometric equation (14) with (15) in the following way.

Theorem 1. *If the system* (2) *is integrable with an additional first integral which is independent of the Hamiltonian and analytic at least in the neighbourhood of the straight-line solution* (4)*, then a particular solution of the Gauss hypergeometric equation* (14) *with* (15) *is given by*

$$\xi = \exp\left[\int \zeta(z)\,\mathrm{d}z\right] \tag{16}$$

where $\zeta(z)$ *is a solution of the algebraic equation*

$$f_0(z)\zeta^n + f_1(z)\zeta^{n-1} + \cdots + f_{n-1}(z)\zeta + f_n(z) = 0 \tag{17}$$

with polynomials $f_j(z)$*.*

Proof. Suppose that the system (2) is integrable and that there exists an additional first integral $\Phi(\boldsymbol{q}, \boldsymbol{p}) = \text{constant}$. Then, as seen in section 3 of Yoshida [14], we have a non-trivial first integral of the NVE (10),

$$I = D^n\Phi := \left|\left(\boldsymbol{\xi}\cdot\frac{\partial}{\partial \boldsymbol{q}} + \boldsymbol{\eta}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)^n \Phi(\boldsymbol{q}, \boldsymbol{p})\right|_{q=cQ,p=cP} = \text{constant} \tag{18}$$

with an integer $n \geqslant 1$ (see [4, 12] for the further details). We can rewrite this first integral in the form

$$I = I(Q, P, \xi, \eta) = \text{constant} \tag{19}$$

which has the following three properties.

Property 1. I is a homogeneous polynomial in the variables (ξ, η).
Property 2. I is a homogeneous polynomial in the weighted degree.
Property 3. I is even or odd in $(P^{m/2}, \eta^{m/2})$.

Property 1 is obvious from equation (18). See appendix A for the proof of properties 2 and 3. From (11)–(13), we have the transformation

$$\begin{aligned} Q &= z^{1/k} \qquad & P &= \left[\frac{m}{k}(1-z)\right]^{1/m} \\ \xi &= \xi \qquad & \eta &= \frac{k}{\mu} z^{(k-1)/k}\left[\frac{m}{k}(1-z)\right]^{1/m}\frac{\mathrm{d}\xi}{\mathrm{d}z} \end{aligned} \tag{20}$$

which transforms (19) into the first integral of the Gauss hypergeometric equation (14) with (15) of the form $\tilde{I}(\mathrm{d}\xi/\mathrm{d}z, \xi, z) = \text{constant}$. Then, the first-order ordinary differential equation $\tilde{I}(\mathrm{d}\xi/\mathrm{d}z, \xi, z) = 0$ gives a particular solution of the Gauss hypergeometric equation (14) with (15). By property 2 of the first integral I, the differential equation $\tilde{I} = 0$ can be written in the form

$$f_0(z)\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^n + f_1(z)\xi\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^{n-1} + \cdots + f_{n-1}(z)\xi^{n-1}\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right) + f_n(z)\xi^n = 0. \tag{21}$$

Furthermore, we can assume that $f_j(z)$ is a polynomial of z for all j (see appendix B for details). Now if we introduce a new variable ζ by $\zeta := (\mathrm{d}\xi/\mathrm{d}z)/\xi = \mathrm{d}\log\xi/\mathrm{d}z$, then the differential equation (21) is reduced to the algebraic equation (17). From a solution $\zeta = \zeta(z)$, we find $\log\xi = \int\zeta(z)\,\mathrm{d}z$. We therefore obtain a particular solution (16). This means that a particular solution is obtained by a combination of algebraic functions, quadratures, and exponential functions of the quadratures. Then we have completed the proof of theorem 1. □

4. Necessary condition for integrability

The quantities $\hat{\lambda} = 1 - \gamma$, $\hat{\mu} = \gamma - \alpha - \beta$, $\hat{\nu} = \beta - \alpha$ are called the differences of exponents at the regular singular points ($z = 0, 1, \infty$) of the Gauss hypergeometric equation (14) (see section 5 of Yoshida [14] for details). Then the following theorem due to Kimura holds.

Theorem 2 (Kimura [5]). *A necessary and sufficient condition for the Gauss hypergeometric equation* (14) *to be solvable in the sense of the differential Galois theory is either:*

- *at least one of* $\hat{\lambda} \pm \hat{\mu} \pm \hat{\nu}$ *is an odd integer, or*
- $\pm\hat{\lambda}, \pm\hat{\mu}, \pm\hat{\nu}$ *take values in table 1, called the table of Schwarz–Hukuhara–Ohasi, in an arbitrary order with integers* l, m, n.

Table 1. Table of Schwarz–Hukuhara–Ohasi.

Case	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\nu}$	
1	$\frac{1}{2}+l$	$\frac{1}{2}+m$	arbitrary	
2	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	
3	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$ even
4	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{4}+n$	
5	$\frac{2}{3}+l$	$\frac{1}{4}+m$	$\frac{1}{4}+n$	$l+m+n=$ even
6	$\frac{1}{2}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	
7	$\frac{2}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{3}+n$	$l+m+n=$ even
8	$\frac{2}{3}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$ even
9	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$ even
10	$\frac{3}{5}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$l+m+n=$ even
11	$\frac{2}{5}+l$	$\frac{2}{5}+m$	$\frac{2}{5}+n$	$l+m+n=$ even
12	$\frac{2}{3}+l$	$\frac{1}{3}+m$	$\frac{1}{5}+n$	$l+m+n=$ even
13	$\frac{4}{5}+l$	$\frac{1}{5}+m$	$\frac{1}{5}+n$	$l+m+n=$ even
14	$\frac{1}{2}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n=$ even
15	$\frac{3}{5}+l$	$\frac{2}{5}+m$	$\frac{1}{3}+n$	$l+m+n=$ even

Theorem 1, together with theorem 2, brings a necessary condition for the integrability of the system (2). The differences of exponents of the Gauss hypergeometric equation (14) with (15) are

$$\hat{\lambda} = \frac{1}{k} \qquad \hat{\mu} = 1 - \frac{1}{m} \qquad \hat{\nu} = \frac{\sqrt{(m-k)^2 + 4mk\Lambda}}{mk}. \tag{22}$$

From the first condition of theorem 2, we obtain

$$\Lambda = \{(jm-1)(jk-1)\} \cup \{(jm+1)(jk-1)+1\} \tag{23}$$

for general values of (m, k). Here, j represents an arbitrary integer. Let us proceed to the second condition of theorem 2. We first consider case 1 of table 1. When both m and k are ± 2, arbitrary Λ is compatible with integrability. When either m or k is ± 2, the values of Λ such that $\pm\hat{\nu} = \frac{1}{2} +$ integer, are compatible with integrability. From this, we have

$$\Lambda = \left\{ \frac{(mk+2k-2m)(mk-2k+2m)}{16mk} + \frac{j(j+1)}{4}mk \right\}. \tag{24}$$

Moreover, taking account of the cases 2–10 and 12–14 of table 1, we obtain additional values of Λ. Then we have the following theorem.

Theorem 3. *If the system* (2) *is integrable, then* Λ *must be one of the following discrete values. The values* (23) *are compatible with integrability for general* (m, k). *When either m or k is* ± 2, *the values* (24), *in addition to the values* (23), *are compatible with integrability. When both m and k are* ± 2, *arbitrary* Λ *is compatible with integrability. When* $m = \pm 2$, $k = \pm 3, \pm 4, \pm 5$ *or* $m = \pm 3, \pm 4, \pm 5$, $k = \pm 2$, *the following values of* Λ, *in addition to the values* (23) *and* (24), *are compatible with integrability.*

•$(m, k) = (\pm 2, \pm 3), (\pm 3, \pm 2)$:

$$\Lambda = \left\{-\tfrac{1}{24} + \tfrac{1}{24}(2+6j)^2\right\} \cup \left\{-\tfrac{1}{24} + \tfrac{1}{24}\left(\tfrac{3}{2}+6j\right)^2\right\} \cup \left\{-\tfrac{1}{24} + \tfrac{1}{24}\left(\tfrac{6}{5}+6j\right)^2\right\} \cup \left\{-\tfrac{1}{24} + \tfrac{1}{24}\left(\tfrac{12}{5}+6j\right)^2\right\} \tag{25}$$

•$(m, k) = (\mp 2, \pm 3), (\pm 3, \mp 2)$:

$$\Lambda = \left\{\tfrac{25}{24} - \tfrac{1}{24}(2+6j)^2\right\} \cup \left\{\tfrac{25}{24} - \tfrac{1}{24}\left(\tfrac{3}{2}+6j\right)^2\right\} \cup \left\{\tfrac{25}{24} - \tfrac{1}{24}\left(\tfrac{6}{5}+6j\right)^2\right\} \cup \left\{\tfrac{25}{24} - \tfrac{1}{24}\left(\tfrac{12}{5}+6j\right)^2\right\} \tag{26}$$

•$(m, k) = (\pm 2, \pm 4), (\pm 4, \pm 2)$:

$$\Lambda = \left\{-\tfrac{4}{32} + \tfrac{1}{32}\left(\tfrac{8}{3}+8j\right)^2\right\} \tag{27}$$

•$(m, k) = (\mp 2, \pm 4), (\pm 4, \mp 2)$:

$$\Lambda = \left\{\tfrac{36}{32} - \tfrac{1}{32}\left(\tfrac{8}{3}+8j\right)^2\right\} \tag{28}$$

•$(m, k) = (\pm 2, \pm 5), (\pm 5, \pm 2)$:

$$\Lambda = \left\{-\tfrac{9}{40} + \tfrac{1}{40}\left(\tfrac{10}{3}+10j\right)^2\right\} \cup \left\{-\tfrac{9}{40} + \tfrac{1}{40}(4+10j)^2\right\} \tag{29}$$

•$(m, k) = (\mp 2, \pm 5), (\pm 5, \mp 2)$:

$$\Lambda = \left\{\tfrac{49}{40} - \tfrac{1}{40}\left(\tfrac{10}{3}+10j\right)^2\right\} \cup \left\{\tfrac{49}{40} - \tfrac{1}{40}(4+10j)^2\right\}. \tag{30}$$

Moreover, when $m, k = \pm 3, \pm 4, \pm 5$, *the following values of* Λ, *in addition to the values* (23), *are compatible with integrability.*

•$(m, k) = (\pm 3, \pm 3)$:

$$\Lambda = \left\{\tfrac{1}{36}\left(\tfrac{9}{2}+9j\right)^2\right\} \cup \left\{\tfrac{1}{36}(15+18j)^2\right\} \cup \left\{\tfrac{1}{36}(3+18j)^2\right\} \cup \left\{\tfrac{1}{36}\left(\tfrac{63}{5}+18j\right)^2\right\} \cup \left\{\tfrac{1}{36}\left(\tfrac{9}{5}+18j\right)^2\right\} \tag{31}$$

•$(m, k) = (\pm 3, \mp 3)$:

$$\Lambda = \left\{1 - \tfrac{1}{36}\left(\tfrac{9}{2}+9j\right)^2\right\} \cup \left\{1 - \tfrac{1}{36}(15+18j)^2\right\} \cup \left\{1 - \tfrac{1}{36}(3+18j)^2\right\} \cup \left\{1 - \tfrac{1}{36}\left(\tfrac{63}{5}+18j\right)^2\right\} \cup \left\{1 - \tfrac{1}{36}\left(\tfrac{9}{5}+18j\right)^2\right\} \tag{32}$$

•$(m, k) = (\pm 3, \pm 4), (\pm 4, \pm 3)$:

$$\Lambda = \left\{-\tfrac{1}{48} + \tfrac{1}{48}(6+12j)^2\right\} \cup \left\{-\tfrac{1}{48} + \tfrac{1}{48}(3+24j)^2\right\} \tag{33}$$

•$(m, k) = (\mp 3, \pm 4), (\pm 4, \mp 3)$:

$$\Lambda = \left\{\tfrac{49}{48} - \tfrac{1}{48}(6 + 12j)^2\right\} \cup \left\{\tfrac{49}{48} - \tfrac{1}{48}(3 + 24j)^2\right\} \tag{34}$$

•$(m, k) = (\pm 3, \pm 5), (\pm 5, \pm 3)$:

$$\Lambda = \left\{-\tfrac{4}{60} + \tfrac{1}{60}\left(\tfrac{15}{2} + 15j\right)^2\right\} \cup \left\{-\tfrac{4}{60} + \tfrac{1}{60}(3 + 30j)^2\right\} \cup \left\{-\tfrac{4}{60} + \tfrac{1}{60}(24 + 30j)^2\right\}$$
$$\cup \left\{-\tfrac{4}{60} + \tfrac{1}{60}(25 + 30j)^2\right\} \cup \left\{-\tfrac{4}{60} + \tfrac{1}{60}(5 + 30j)^2\right\} \tag{35}$$

•$(m, k) = (\mp 3, \pm 5), (\pm 5, \mp 3)$:

$$\Lambda = \left\{\tfrac{64}{60} - \tfrac{1}{60}\left(\tfrac{15}{2} + 15j\right)^2\right\} \cup \left\{\tfrac{64}{60} - \tfrac{1}{60}(3 + 30j)^2\right\} \cup \left\{\tfrac{64}{60} - \tfrac{1}{60}(24 + 30j)^2\right\}$$
$$\cup \left\{\tfrac{64}{60} - \tfrac{1}{60}(25 + 30j)^2\right\} \cup \left\{\tfrac{64}{60} - \tfrac{1}{60}(5 + 30j)^2\right\} \tag{36}$$

•$(m, k) = (\pm 4, \pm 4)$:

$$\Lambda = \left\{\tfrac{1}{64}\left(\tfrac{80}{3} + 32j\right)^2\right\} \tag{37}$$

•$(m, k) = (\pm 4, \mp 4)$:

$$\Lambda = \left\{1 - \tfrac{1}{64}\left(\tfrac{80}{3} + 32j\right)^2\right\} \tag{38}$$

•$(m, k) = (\pm 5, \pm 5)$:

$$\Lambda = \left\{\tfrac{1}{100}\left(\tfrac{125}{3} + 50j\right)^2\right\} \cup \left\{\tfrac{1}{100}(45 + 50j)^2\right\} \cup \left\{\tfrac{1}{100}(5 + 50j)^2\right\} \tag{39}$$

•$(m, k) = (\pm 5, \mp 5)$:

$$\Lambda = \left\{1 - \tfrac{1}{100}\left(\tfrac{125}{3} + 50j\right)^2\right\} \cup \left\{1 - \tfrac{1}{100}(45 + 50j)^2\right\} \cup \left\{1 - \tfrac{1}{100}(5 + 50j)^2\right\}. \tag{40}$$

When $m = 2$, the statement of theorem 3 corresponds to that of the theorem due to Morales-Ruiz and Ramis [7].

5. Examples

Let us take the Hamiltonian system

$$H = \frac{(p_1 + p_2)^m + (p_1 - p_2)^m}{2m} + \frac{(q_1 + q_2)^k + (q_1 - q_2)^k}{2k} \tag{41}$$

as an example to confirm the statement of theorem 1. The system (41) is integrable with an additional first integral

$$\Phi = \frac{(p_1 + p_2)^m - (p_1 - p_2)^m}{2m} + \frac{(q_1 + q_2)^k - (q_1 - q_2)^k}{2k} = \text{constant}. \tag{42}$$

The parameter Λ equals $(m-1)(k-1)$ for the straight-line solution (4) with $(c_1, c_2) = (1, 0)$. The first integral (19) of the NVE is $I = \xi Q^{k-1} + \eta P^{k-1} = \text{constant}$, which yields the first integral of the Gauss hypergeometric equation, $\tilde{I} = z^{(k-1)/k}[(m-1)\xi + m(1-z)\,\mathrm{d}\xi/\mathrm{d}z] = \text{constant}$. From $\tilde{I} = 0$, we obtain $\xi = \exp[\int (m-1)\,\mathrm{d}z/m(z-1)] = (z-1)^{(m-1)/m}$. This indeed satisfies the Gauss hypergeometric equation (14) with $\Lambda = (m-1)(k-1)$. We can see that the parameter $\Lambda = (m-1)(k-1)$ is an example of (23) with $j = 1$.

Next, we take the Hamiltonian system of the form

$$H = \frac{1}{m}(p_1^2 + p_2^2)^{m/2} + V_k(q_1, q_2) \tag{43}$$

where m is a positive even integer and $V_k(q_1, q_2)$ is a homogeneous polynomial of degree k. When $m = 2$, this system is reduced to the system (1). We first consider the case where

$$V_k(q_1, q_2) = \frac{1}{k}(q_1^2 + q_2^2)^{k/2}. \tag{44}$$

The system (43) with (44) is integrable with an additional first integral $\Phi = q_1 p_2 - q_2 p_1$. The parameter Λ equals 1 for the straight-line solution (4) with $(c_1, c_2) = (0, 1)$. The first integral (19) of the NVE is $I = D\Phi = \xi P - \eta Q =$ constant, which yields the first integral of the Gauss hypergeometric equation, $\tilde{I} = (m/k)^{1/m}(1-z)^{1/m}(\xi - kz\,\mathrm{d}\xi/\mathrm{d}z) =$ constant. From $\tilde{I} = 0$, we obtain $\xi = \exp[\int(1/kz)\,\mathrm{d}z] = z^{1/k}$. This indeed satisfies the Gauss hypergeometric equation (14) with $\Lambda = 1$. We can see that the parameter $\Lambda = 1$ is an example of (23) with $j = 0$. Next we take the case where

$$V_k(q_1, q_2) = \frac{1}{kr}\left[\left(\frac{r+q_2}{2}\right)^{k+1} + (-1)^k\left(\frac{r-q_2}{2}\right)^{k+1}\right] \qquad r = \sqrt{q_1^2 + q_2^2} \tag{45}$$

as an example to see how theorem 3 works. The parameter Λ equals $(k-1)/2k$ for the straight-line solution (4) with $(c_1, c_2) = (0, 1)$. By comparing $(k-1)/2k$ with the discrete values Λ listed in theorem 3, we find that the cases where $(m, k) = (2, \text{arbitrary}), (\text{arbitrary}, 1)$ are only compatible with integrability. When $m = 2$, the parameter Λ corresponds to (24) with $j = 0$ for arbitrary k, and the system (43) is integrable with an additional first integral $\Phi = p_1(q_1 p_2 - q_2 p_1) + (k-1)q_1^2 V_{k-1}/2k$ [9]. When $k = 1$, the parameter Λ becomes 0, which is an example of (23) with $j = 0$. Then we have $V_1(q_1, q_2) = q_2$ and find it integrable with an additional first integral $\Phi = p_1$. Finally, we take the case where

$$V_k(q_1, q_2) = \frac{1}{k}(q_1^k + q_2^k). \tag{46}$$

When $m = 2$, the system (43) with (46) is obviously integrable with an additional first integral $\Phi = \frac{1}{2}p_1^2 + (1/k)q_1^k$. Let us consider the case where $m \neq 2$. When $k \geqslant 2$, the parameter Λ equals 0 for the straight-line solution (4) with $(c_1, c_2) = (0, 1)$, which is compatible with integrability. For $k = 1$, there exist no straight-line solutions and theorem 3 is not applicable. However, the case where $k = 1$ is integrable with an additional first integral $\Phi = p_1 - p_2$. For $k = 6k' + 2$ with a non-negative integer k', there exists another straight-line solution (4) with (c_1, c_2) satisfying $(c_1^2 + c_2^2)^{m/2-1} = c_1^{k-2} = c_2^{k-2} = 1$. This straight-line solution yields the parameter $\Lambda = k - 1 = 6k' + 1$, which is compatible with integrability when and only when $k' = 0$, i.e. $k = 2$. This means that the cases where $k = 6k' + 2$ with a positive integer k' are non-integrable. When $k = 2$, the system (43) with (46) is integrable with the additional first integral $\Phi = q_1 p_2 - q_2 p_1$. It remains to be seen whether the other cases are integrable or non-integrable.

All of the above examples are separable cases. As for non-separable cases, the authors do not know of any examples except the case for $m = 2$, where some non-separable examples are known [14].

6. Necessary condition in terms of Kowalevski exponents

Let a system of differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = F_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, N) \tag{47}$$

be invariant by the scale transformation (a scale-invariant system)

$$t \to \alpha^{-1}t \qquad x_i \to \alpha^{g_i}x_i. \qquad (i = 1, 2, \ldots, N). \tag{48}$$

Eigenvalues of an $N \times N$ matrix K whose elements are given by

$$K_{ij} = \left(\frac{\partial F_i}{\partial x_j}\right)_{\boldsymbol{x}=\boldsymbol{d}} + \delta_{ij} g_i \tag{49}$$

are called the Kowalevski exponents (KE) [11]. Here, $\boldsymbol{d} = (d_1, d_2, \ldots, d_N)$ is a solution of the algebraic equation

$$F_i(d_1, d_2, \ldots, d_N) = -g_i d_i \qquad (i = 1, 2, \ldots, N). \tag{50}$$

A concise review of the KE is given in [1]. Since the system (2) is invariant by the scale transformation

$$t \to \alpha^{-1} t \qquad \boldsymbol{q} \to \alpha^{g} \boldsymbol{q} \qquad \boldsymbol{p} \to \alpha^{g'} \boldsymbol{p} \qquad g = \frac{m}{mk - m - k} \qquad g' = \frac{g+1}{m-1} \tag{51}$$

we can consider the KE for the system (2). The characteristic equation, which determines the KE, is

$$\det(\rho I_4 - K) = \det\left[(\rho - g)(\rho - g') I_4 + \partial^2 T(\boldsymbol{d}') \partial^2 V(\boldsymbol{d})\right] = 0 \tag{52}$$

where $\boldsymbol{d}$ and $\boldsymbol{d}'$ satisfy

$$\nabla T(\boldsymbol{d}') = -g \boldsymbol{d} \qquad \nabla V(\boldsymbol{d}) = g' \boldsymbol{d}. \tag{53}$$

Let ν_i and τ_i denote the eigenvalues of Hessian matrices $\partial^2 T(\boldsymbol{d}')$ and $\partial^2 V(\boldsymbol{d})$, respectively. Then, we see that the KE are solutions of

$$(\rho - g)(\rho - g') + \nu_i \tau_i = 0. \tag{54}$$

From (5) and (53), we have the relations among $\boldsymbol{c}$, $\boldsymbol{d}$ and $\boldsymbol{d}'$, given by

$$\boldsymbol{d} = \{-g(g')^{m-1}\}^{g/m} \boldsymbol{c} \qquad \boldsymbol{d}' = \{g'(-g)^{k-1}\}^{g'/k} \boldsymbol{c}. \tag{55}$$

This means

$$\begin{aligned} \partial^2 T(\boldsymbol{d}') &= \{g'(-g)^{k-1}\}^{(m-2)g/m} \partial^2 T(\boldsymbol{c}) \\ \partial^2 V(\boldsymbol{d}) &= \{-g(g')^{m-1}\}^{(k-2)g'/k} \partial^2 V(\boldsymbol{c}). \end{aligned} \tag{56}$$

Therefore, we have

$$\nu_i = \{g'(-g)^{k-1}\}^{(m-2)g/m} \mu_i \qquad \tau_i = \{-g(g')^{m-1}\}^{(k-2)g'/k} \lambda_i. \tag{57}$$

Here, μ_i and λ_i are eigenvalues of the Hessian matrices $\partial^2 T(\boldsymbol{c})$ and $\partial^2 V(\boldsymbol{c})$, respectively ($\mu_1 = \mu, \lambda_1 = \lambda, \mu_2 = m - 1, \lambda_2 = k - 1$). Then, equation (54) becomes

$$(\rho - g)(\rho - g') - g g' \mu_i \lambda_i = 0 \qquad (i = 1, 2) \tag{58}$$

which yields two pairs of KE. For $i = 2$, we have a pair of KE

$$(\rho_3, \rho_4) = \left(\frac{mk}{mk - m - k}, -1\right). \tag{59}$$

For $i = 1$, equation (58) becomes

$$\rho^2 - (g + g')\rho + g g'(1 - \Lambda) = 0 \tag{60}$$

from which we obtain the other pair of KE for the values of Λ listed in theorem 3. For $\Lambda = (jm + 1)(jk - 1) + 1$, the pair of KE is

$$(\rho_1, \rho_2) = \left(\frac{k + jmk}{mk - m - k}, \frac{m - jmk}{mk - m - k}\right). \tag{61}$$

For $\Lambda = (jm-1)(jk-1)$, the pair of KE is

$$(\rho_1, \rho_2) = \left(\frac{jmk}{mk-m-k}, \frac{m+k-jmk}{mk-m-k}\right). \tag{62}$$

For the values of Λ given by (24), the pair of KE is

$$(\rho_1, \rho_2) = \left(\frac{2(m+k)+(2j+1)mk}{4(mk-m-k)}, \frac{2(m+k)-(2j+1)mk}{4(mk-m-k)}\right). \tag{63}$$

For $\Lambda = -(m-k)^2/4mk + X^2/4mk$ listed in (25)–(40), the pair of KE is

$$(\rho_1, \rho_2) = \left(\frac{m+k+X}{2(mk-m-k)}, \frac{m+k-X}{2(mk-m-k)}\right). \tag{64}$$

We can see that all the KE listed above are rational numbers. Then we obtain the following corollary of theorem 3.

Corollary 1. *If the system* (2) *is integrable, then all the KE are specific rational numbers listed above.*

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Appendix A

We give the proof of properties 2 and 3 of the first integral I mentioned in section 3.

Proof. A function $\Phi(x_1, x_2, \ldots, x_N)$ is said to be homogeneous in the weighted degree with a weight M if the scale transformation (48) multiplies Φ by α^M, i.e.

$$\Phi(\alpha^{g_1}x_1, \alpha^{g_2}x_2, \ldots, \alpha^{g_N}x_N) = \alpha^M \Phi(x_1, x_2, \ldots, x_N). \tag{A1}$$

As seen in appendix A of Yoshida [14], we can assume that a first integral of scale-invariant systems is a homogeneous polynomial in the weighted degree. The NVE (10) are invariant by the scale transformation

$$t \to \alpha^{-1}t \qquad Q \to \alpha^g Q \qquad P \to \alpha^{g'} P \qquad \xi \to \alpha^{g'}\xi \qquad \eta \to \alpha^{2g'-g}\eta \tag{A2}$$

with constants g and g' given in section 5. Therefore, we can assume that the first integral I is a homogeneous polynomial in the weighted degree.

Property 3 can be recognized from a general theory of the time reflection symmetry discussed in section 2.3 of [3]. If we introduce new variables, $\bar{P} := P^{m/2}$, $\bar{\eta} := \eta^{m/2}$, then the NVE (10) become

$$\frac{d\xi}{dt} = \mu \bar{P}^{2(m-2)/m}\bar{\eta}^{2/m} \qquad \frac{2}{m}\bar{\eta}^{2/m-1}\frac{d\bar{\eta}}{dt} = -\lambda Q^{k-2}\xi \tag{A3}$$

which are invariant by the change of variables ('time reflection')

$$t \to (-1)^{2(m-1)/m}t \qquad Q \to Q \qquad \bar{P} \to -\bar{P} \qquad \xi \to \xi \qquad \bar{\eta} \to -\bar{\eta}. \tag{A4}$$

Suppose that $I(Q, \bar{P}, \xi, \bar{\eta})$ is a first integral of the system (A3). Then $I(Q, -\bar{P}, \xi, -\bar{\eta})$ is also a first integral because of the 'time reflection' symmetry of the system (A3). Thus, I^+ and I^- given by

$$I^+ = \frac{I(Q, \bar{P}, \xi, \bar{\eta}) + I(Q, -\bar{P}, \xi, -\bar{\eta})}{2} \qquad I^- = \frac{I(Q, \bar{P}, \xi, \bar{\eta}) - I(Q, -\bar{P}, \xi, -\bar{\eta})}{2} \tag{A5}$$

are also first integrals. Every first integral I can be decomposed as $I = I^+ + I^-$, where I^+ is even in the variables $(\bar{P}, \bar{\eta})$ and I^- is odd. Therefore, the first integral I can be assumed to be even or odd in $(P^{m/2}, \eta^{m/2})$ from the beginning. □

Appendix B

We prove that all $f_j(z)$ in (21) can be assumed to be polynomials.

Proof. Let $M = CQ^aP^b\xi^c\eta^d$ and $M' = C'Q^{a'}P^{b'}\xi^{c'}\eta^{d'}$ be any two monomials in the polynomial (19), where C, C' are constants and $a, b, c, d, a', b', c', d'$ are integers. By substituting (20) in M, M', we have

$$M = C\left(\frac{k}{\mu}\right)^d\left(\frac{m}{k}\right)^{(b+d)/m} z^{a/k+(k-1)d/k}(1-z)^{(b+d)/m}\xi^c\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^d \tag{B1}$$

$$M' = C'\left(\frac{k}{\mu}\right)^{d'}\left(\frac{m}{k}\right)^{(b'+d')/m} z^{a'/k+(k-1)d'/k}(1-z)^{(b'+d')/m}\xi^{c'}\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^{d'}. \tag{B2}$$

In order to complete the proof, it is enough to show that the differences of the powers of z, $(1-z)$ in M, M', given by

$$\Delta_1 = \frac{1}{k}a + \frac{k-1}{k}d - \left(\frac{1}{k}a' + \frac{k-1}{k}d'\right) \qquad \Delta_2 = \frac{1}{m}b + \frac{1}{m}d - \left(\frac{1}{m}b' + \frac{1}{m}d'\right) \tag{B3}$$

are both integers. By property 1 of the first integral I, we can put

$$c - c' = -n \qquad d - d' = n \tag{B4}$$

with an integer n. By property 2 of the first integral I, the relation

$$ga + g'(b+c) + (2g'-g)d = ga' + g'(b'+c') + (2g'-g)d' \tag{B5}$$

holds with constants g and g'. From (B5) together with (B4), we have

$$a - a' = -\frac{k}{m}(b-b') - \frac{k-m}{m}n. \tag{B6}$$

Then Δ_1, Δ_2 are reduced to

$$\Delta_1 = \frac{1}{m}\{b' - b + (m-1)n\} \qquad \Delta_2 = \frac{1}{m}(b - b' + n). \tag{B7}$$

By property 3 of the first integral I, we can always put either

$$\frac{2b}{m} + \frac{2d}{m} = 2s \qquad \frac{2b'}{m} + \frac{2d'}{m} = 2s'$$

or (B8)

$$\frac{2b}{m} + \frac{2d}{m} = 2s+1 \qquad \frac{2b'}{m} + \frac{2d'}{m} = 2s'+1$$

with integers s, s'. In both cases, we have $b - b' = -n + m(s-s')$ and then Δ_1, Δ_2 are reduced to

$$\Delta_1 = s' - s + n \qquad \Delta_2 = s - s' \tag{B9}$$

which are both integers. Then we have completed the proof. □

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